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A New Method for Predicting the Solar Heat Gain of Complex Fenestration Systems II. Detailed Description of the Matrix Layer Calculation

J. H. Klems, Ph.D.

Building Technologies Program
Energy and Environment Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

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A New Method for Predicting the Solar Heat Gain of Complex Fenestration Systems
II. Detailed Description of the Matrix Layer Calculation

J. H. Klems, Ph.D.

Abstract

A new method of predicting the solar heat gain through complex fenestration systems involving nonspecular layers such as shades or blinds has been examined in a project jointly sponsored by ASHRAE and DOE. In this method, a scanning radiometer is used to measure the bi-directional radiative transmittance and reflectance of each layer of a fenestration system. The properties of systems containing these layers are then built up computationally from the measured layer properties using a transmission/multiple-reflection calculation. The calculation produces the total directional-hemispherical transmittance of the fenestration system and the layer-by-layer absorptances. These properties are in turn combined with layer-specific measurements of the inward-flowing fractions of absorbed solar energy to produce the overall solar heat gain coefficient.

A preceding paper outlined the method and provided the physical derivation of the calculation. In this second of a series of related papers the detailed development of the matrix layer calculation is presented.

Introduction

This paper continues the summary of the findings of a research project, jointly sponsored by ASHRAE and DOE, to develop a method of determining the solar heat gain through complex fenestration systems, by which is meant systems including one or more non specular layers (i.e., layers that scatter radiation in many directions). A window with a drape, shade, or venetian blind is a familiar example of such a fenestration system. In a preceding paper (Klems 1993A), (which will be denoted Paper I) this project was outlined and a matrix calculation method for computing the properties of multilayer systems from those of individual layers was developed. A number of relations were asserted without proof. In this paper the matrix calculation method is discussed in detail and those assertions are substantiated.

In Paper I the concept of solar heat gain coefficient is extended for multilayer non-specular systems into a direction-dependent quantity

$$F(\theta,\phi) = T_{fH}(\theta,\phi) + \sum_{i=1}^{M} N_i A_{fi}(\theta,\phi), \qquad (1.1)$$

where T_{fH} denotes the *front directional-hemispherical transmittance of the system* , and N_i, A_{fi} denote the inward-flowing fraction and front absorptance, respectively, of the i^{th} *layer*. The *layer inward-flowing-fraction*, N_i, represents the fraction of the energy absorbed in the i^{th} layer that ultimately flows into the building space. In order to calculate T_{fH} and A_{fi}, that paper began by defining the wavelength-averaged solar-optical properties of the i^{th} layer in a fenestration in terms of its bi-directional transmittance and reflectance distribution functions (Nicodemus 1965) as follows:

$$I(\theta_i,\phi_i) = \tau_i^f(\theta_i,\phi_i;\theta_{i-1},\phi_{i-1})E(\theta_{i-1},\phi_{i-1}), \tag{1.2a}$$

where (θ_i,ϕ_i) represents the outgoing direction of the radiation, $(\theta_{i-1},\phi_{i-1})$ the incoming direction, and $E(\theta_{i-1},\phi_{i-1})$ is the irradiance (energy per unit area) incident on the front surface of the layer by radiation going in the incident direction in the +z hemisphere. The quantity τ_i^f is the front bi-directional transmittance distribution function of the layer, and $I(\theta_i,\phi_i)$ is the radiance (energy per unit area per unit solid angle) of the radiation emerging out of the back side of the layer in the outgoing direction, which is in the +z hemisphere. In the coordinate system shown in Figure 1, the z axis is the outward normal to the back side of the layer. Diagrams will be drawn with the front sides of layers on the left, so that radiation into the +z hemisphere may sometimes be referred to as right-moving. Layers are numbered from front to rear. Reflectance from the front side of the layer produces an outgoing radiance, denoted by J,

$$J(\theta_i^r,\phi_i^r) = \rho_i^f(\theta_i^r,\phi_i^r;\theta_{i-1},\phi_{i-1})E(\theta_{i-1},\phi_{i-1}) \tag{1.2b}$$

consisting of radiation in the reflected direction (θ_i^r,ϕ_i^r), which is in the -z ("left-moving" or "backward") hemisphere, where ρ_i^f is the front reflectance distribution function for layer i. Since the layer cannot be assumed to be front-back symmetric, there are analogous relations for radiation incident on the back side:

$$J(\theta_i^r,\phi_i^r) = \tau_i^b(\theta_i^r,\phi_i^r;\theta_{i+1}^r,\phi_{i+1}^r)E^r(\theta_{i+1}^r,\phi_{i+1}^r), \tag{1.2c}$$

$$I(\theta_i,\phi_i) = \rho_i^b(\theta_i,\phi_i;\theta_{i+1}^r,\phi_{i+1}^r)E^r(\theta_{i+1}^r,\phi_{i+1}^r), \tag{1.2d}$$

where E^r denotes the back-side irradiance from left-moving radiation in the direction $(\theta_{i+1}^r,\phi_{i+1}^r)$ (the subscript denoting that this radiation comes from the i+1st layer), and τ_i^b, ρ_i^b are the back transmittance and reflectance distribution functions, respectively, of the ith layer. The incident irradiance may be calculated from the radiance emerging from the adjacent layers as follows:

$$dE(\theta_{i-1},\phi_{i-1}) = I(\theta_{i-1},\phi_{i-1})\cos(\theta_{i-1})d\Omega_{i-1}, \tag{1.3a}$$

$$dE^r(\theta_{i+1}^r,\phi_{i+1}^r) = J(\theta_{i+1}^r,\phi_{i+1}^r)\cos(\theta_{i+1}^r)d\Omega_{i+1}^r, \tag{1.3b}$$

where $d\Omega_{i-1} = \sin(\theta_{i-1})d\theta_{i-1}d\phi_{i-1}$ and similarly for $d\Omega_{i+1}^r$. The equation for calculating the irradiance emerging from a pair of layers (*without* considering interreflectances between layers) was shown in Paper I to be

$$I(\theta_2,\phi_2) = \int d\Omega_1 \cos(\theta_1)\tau_2^f(\theta_2,\phi_2;\theta_1,\phi_1)\tau_1^f(\theta_1,\phi_1;\theta_0,\phi_0)E(\theta_0,\phi_0). \tag{1.4}$$

Let us now proceed to develop this into a multilayer calculation method for determining the system transmission and absorption in equation 1.1.

Derivation of the Matrix Method of Calculating Fenestration System Optical Properties

We begin with the transmission through a two-layer system, neglecting interreflections between layers, as given by equation 1.4. If we compare this equation with 1.2a, we see that in effect the equation defines the effective front transmission of a two-layer system in terms of the properties of the individual layers:

$$T^{f}_{2,\{1,2\}}(\theta_2,\phi_2;\theta_0,\phi_0)=\int d\Omega_1\cos(\theta_1)\tau^{f}_2(\theta_2,\phi_2;\theta_1,\phi_1)\tau^{f}_1(\theta_1,\phi_1;\theta_0,\phi_0), \tag{2.1}$$

where the symbol $T^{f}_{2,\{1,2\}}(\theta_2,\phi_2;\theta_0,\phi_0)$ denotes a (sub-) system front transmittance. The first number in the subscript indicates that it is a two-layer transmittance, with the numbers inside the curly brackets indicating that it begins with layer 1 and ends with layer 2. These numbers may be omitted it there is no ambiguity possible, i.e., if one is including all the layers in a given system. We stress that equation 2.1 is only a provisional expression for $T^{f}_{2,\{1,2\}}$, since it does not include layer interreflections.

We next rewrite equation 2.1 as a sum of a finite number of terms by breaking the solid angle integration into a sum over finite elements $\Delta\Omega^{l}_1$. Corresponding to each element of solid angle, there is a direction $(\theta^{l}_1,\phi^{l}_1)$ within the solid angle such that the value of the bi-directional transmittance distribution function at that set of angles is equal to its mean value over the solid angle.* (This follows from the Mean Value Theorem of calculus.) If we define similar solid angle elements and corresponding directions $(\theta^{k}_0,\phi^{k}_0)$ and $(\theta^{m}_2,\phi^{m}_2)$ for the incident and outgoing angles, respectively, then equation 2.1 becomes

$$T^{f}_{2,\{1,2\}}(\theta^{m}_2,\phi^{m}_2;\theta^{k}_0,\phi^{k}_0)=\sum_{l}\tau^{f}_2(\theta^{m}_2,\phi^{m}_2;\theta^{l}_1,\phi^{l}_1)\Delta\Omega^{l}_1\cos(\theta^{l}_1)\tau^{f}_1(\theta^{l}_1,\phi^{l}_1;\theta^{k}_0,\phi^{k}_0), \tag{2.2}$$

where the τ^{f}_i are now the front biconical transmittances of the layers. As pointed out by Winkelmann (Papamichael and Winkelmann 1986), this equation is very suggestive of a matrix multiplication, which can be seen if we suppress the explicit angle dependence and relabel the transmittance with a pair of subscripts corresponding to the incident and outgoing directions:

$$\left(T^{f}_{2,\{1,2\}}\right)_{mk}=\sum_{l}\left(\tau^{f}_2\right)_{ml}\Delta\Omega^{l}_1\cos(\theta^{l}_1)\left(\tau^{f}_1\right)_{lk}. \tag{2.3}$$

We therefore make the following definitions. We choose a specific ordering, shown in Figure 2, for the solid angle elements ($l=1,\ldots N$) and arrange the corresponding incoming irradiances for layer i in an N-element column vector,

* Since the mean value of the transmittance over the finite solid angle element is the biconical transmittance (provided that one makes a similar finite solid angle for the incident radiation), the bidirectional transmittance at the chosen pair of angles is equal to the biconical transmittance, and similarly for reflectance. Hence, the modifiers bidirectional and biconical will be used essentially interchangeably in the text.

$$\mathbf{E}_{i-1} = \begin{Bmatrix} E(\theta_{i-1}^1, \phi_{i-1}^1) \\ E(\theta_{i-1}^2, \phi_{i-1}^2) \\ \dots \\ E(\theta_{i-1}^N, \phi_{i-1}^N) \end{Bmatrix}, \tag{2.4}$$

(since by our labeling convention the incoming radiation for layer i carries the label i-1), and the outgoing radiances in a similar column vector,

$$\mathbf{I}_i = \begin{Bmatrix} I(\theta_i^1, \phi_i^1) \\ I(\theta_i^2, \phi_i^2) \\ \dots \\ I(\theta_i^N, \phi_i^N) \end{Bmatrix}. \tag{2.5}$$

We then define a matrix of bi-directional (front) transmittances for layer i by

$$\boldsymbol{\tau}_i^f = \begin{Bmatrix} \tau_i^f(\theta_i^1, \phi_i^1; \theta_{i-1}^1, \phi_{i-1}^1) & \tau_i^f(\theta_i^1, \phi_i^1; \theta_{i-1}^2, \phi_{i-1}^2) & \dots & \tau_i^f(\theta_i^1, \phi_i^1; \theta_{i-1}^N, \phi_{i-1}^N) \\ \tau_i^f(\theta_i^2, \phi_i^2; \theta_{i-1}^1, \phi_{i-1}^1) & \tau_i^f(\theta_i^2, \phi_i^2; \theta_{i-1}^2, \phi_{i-1}^2) & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \tau_i^f(\theta_i^N, \phi_i^N; \theta_{i-1}^1, \phi_{i-1}^1) & \dots & \dots & \tau_i^f(\theta_i^N, \phi_i^N; \theta_{i-1}^N, \phi_{i-1}^N) \end{Bmatrix}. \tag{2.6}$$

It can easily be seen that with this notation the analog of the equation 1.4a relating radiance, irradiance and (bi-directional) transmittance distribution functions is a matrix equation,

$$\mathbf{I}_i = \boldsymbol{\tau}_i^f \cdot \mathbf{E}_{i-1}. \tag{2.7}$$

Only the presence of the factor $\Delta\Omega_1^l \cos(\theta_1^l)$ prevents equation 2.3 from having the form of a matrix multiplication. The function of this factor in the equation is to convert the outgoing radiance of layer 1 for the lth solid angle element, $I(\theta_1^l, \phi_1^l)$, into the incoming irradiance on the front surface of layer 2 for the same solid angle element, $E(\theta_1^l, \phi_1^l)$. Physically, one can say that this factor propagates the radiation along its direction from layer 1 to layer 2 as it converts from radiance to irradiance. We therefore define a diagonal propagation matrix (layer i to layer i+1),

$$\boldsymbol{\Lambda}_i = \begin{Bmatrix} \Delta\Omega_i^1 \cos(\theta_i^1) & 0 & \dots & 0 \\ 0 & \Delta\Omega_i^2 \cos(\theta_i^2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta\Omega_i^N \cos(\theta_i^N) \end{Bmatrix}. \tag{2.8}$$

Then

$$\mathbf{E}_{i-1} = \boldsymbol{\Lambda}_{i-1} \cdot \mathbf{I}_{i-1} \tag{2.9}$$

and equation 2.3 becomes in the new notation

$$\mathbf{T}^{f}_{2,\{1,2\}} = \boldsymbol{\tau}^{f}_{2} \cdot \boldsymbol{\Lambda}_{1} \cdot \boldsymbol{\tau}^{f}_{1}. \tag{2.10}$$

This process is indicated diagramatically in Figure 3(A), together with an abbreviated diagram in 3(B) that will be used to represent it. Note that in this notation matrices appear from right to left in the order in which the ray encounters the corresponding surface, and that the matrices are not commutative.

We next address the problem of extending this calculation to include multiple reflections between layers, as illustrated in Figure 4. In this figure the directly transmitted ray is drawn as a heavy arrow. This would enter a calculation such as equation 2.10. If we denote the direction of this ray as it is incident on layer i by $\theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}$, one can see from the figure that a front reflection by layer i followed by a reflection from the back of layer i-1 can produce additional radiation incident in the same direction, and subsequent reflections of this ray will produce additional incident radiation. All of this additional incident radiation will produce additional transmitted radiation which must be included in the transmission calculation.

Using equation 1.4b, the reflected radiance from the direct ray at the front surface of layer i will be

$$J(\theta^{r}_{i}, \phi^{r}_{i}) = \rho^{f}_{i}(\theta^{r}_{i}, \phi^{r}_{i}; \theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}) E(\theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}), \tag{2.11}$$

which produces an incident irradiance on the back side of layer i-1 (using equation 1.3b) given by

$$dE^{r}(\theta^{r}_{i}, \phi^{r}_{i}) = J(\theta^{r}_{i}, \phi^{r}_{i}) \cos(\theta^{r}_{i}) d\Omega^{r}_{i}, \tag{2.12}$$

and after reflection from the back side of layer i-1 produces an additional incident irradiance on the front side of layer i due to a first order (pair of) reflections of

$$\begin{gathered} dE^{(1)}(\theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}) = \\ \cos(\theta^{(0)}_{i-1}) d\Omega^{(0)}_{i-1} \int \rho^{b}_{i}(\theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}; \theta^{r}_{i}, \phi^{r}_{i}) \rho^{f}_{i}(\theta^{r}_{i}, \phi^{r}_{i}; \theta_{i-1}, \phi_{i-1}) dE(\theta_{i-1}, \phi_{i-1}) \cos(\theta^{r}_{i}) d\Omega^{r}_{i} \end{gathered} \tag{2.13}$$

When we note that for the directly transmitted ray,

$$dE(\theta_{i-1}, \phi_{i-1}) = d\Omega_{i-1} \cos(\theta_{i-1}) \tau^{f}_{i-1}(\theta_{i-1}, \phi_{i-1}; \theta_{i-2}, \phi_{i-2}), \tag{2.14}$$

as in equation 2.1 (where i=2), and that each outgoing ray from layer i-1 may undergo a pair of reflections such that it arrives back at the front side of layer i with a direction $\theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}$, we see that the effect of the first-order reflection is to replace the function $\tau^{f}_{i-1}(\theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}; \theta_{i-2}, \phi_{i-2})$ with the convolution integral

$$\int d\Omega_{i-1} \cos(\theta_{i-1}) M^{(1)}(\theta^{(0)}_{i-1}, \phi^{(0)}_{i-1}; \theta_{i-1}, \phi_{i-1}) \tau^{f}_{i-1}(\theta_{i-1}, \phi_{i-1}; \theta_{i-2}, \phi_{i-2}), \tag{2.15}$$

where the first-order reflection function is given by

$$M^{(1)}(\theta_{i-1}^{(a)},\phi_{i-1}^{(a)};\theta_{i-1}^{(b)},\phi_{i-1}^{(b)}) = \cos(\theta_{i-1}^{(a)})d\Omega_{i-1}^{(a)}\int \rho_i^b(\theta_{i-1}^{(a)},\phi_{i-1}^{(a)};\theta_i^r,\phi_i^r)\rho_i^f(\theta_i^r,\phi_i^r;\theta_{i-1}^{(b)},\phi_{i-1}^{(b)})\cos(\theta_i^r)d\Omega_i^r \cdot \tag{2.16}$$

This function essentially maps an incident direction (b) in zeroth order to an incident direction (a) in first order reflection. By repeating the above argument it is easy to show that an nth order (pair of) reflection(s) produces a convolution integral of the same form as 2.15 with an nth order reflection function $M^{(n)}$ in place of $M^{(1)}$, where $M^{(n)}$ is given by the recursion relation

$$M^{(n+1)}(\theta_{i-1}^{(a)},\phi_{i-1}^{(a)};\theta_{i-1}^{(b)},\phi_{i-1}^{(b)}) = \int M^{(1)}(\theta_{i-1}^{(a)},\phi_{i-1}^{(a)};\theta_{i-1}^{(c)},\phi_{i-1}^{(c)})M^{(n)}(\theta_{i-1}^{(c)},\phi_{i-1}^{(c)};\theta_{i-1}^{(b)},\phi_{i-1}^{(b)})\cos(\theta_{i-1}^{(c)})d\Omega_{i-1}^{(c)} \cdot \tag{2.17}$$

Reflections to all orders are now included in the transmission calculation by replacing the function $\tau_{i-1}^f(\theta_{i-1}^{(0)},\phi_{i-1}^{(0)};\theta_{i-2},\phi_{i-2})$ with the sum

$$\tau_{i-1}^f(\theta_{i-1}^{(0)},\phi_{i-1}^{(0)};\theta_{i-2},\phi_{i-2}) + \sum_{n=1}^{\infty}\int M^{(n)}(\theta_{i-1}^{(0)},\phi_{i-1}^{(0)};\theta_{i-1},\phi_{i-1})\tau_{i-1}^f(\theta_{i-1},\phi_{i-1};\theta_{i-2},\phi_{i-2})\cos(\theta_{i-1})d\Omega_{i-1} \cdot \tag{2.18}$$

We now convert these integrals into finite sums in the same manner as was done in the derivation of equation 2.10. Referring all backward-hemisphere rays to the reflected coordinate system in Fig. 1.2 and breaking up the solid angle for integration into finite pieces in the same manner as was done above, we define reflected irradiance and radiance column vectors by

$$\mathbf{E}_{i+1}^r = \begin{Bmatrix} E_{i+1}^r(\theta_{i+1}^{r,1},\phi_{i+1}^{r,1}) \\ E_{i+1}^r(\theta_{i+1}^{r,2},\phi_{i+1}^{r,2}) \\ \dots \\ E_{i+1}^r(\theta_{i+1}^{r,N},\phi_{i+1}^{r,N}) \end{Bmatrix} \tag{2.19}$$

and

$$J_{i+1} = \begin{Bmatrix} J(\theta_{i+1}^{r,1},\phi_{i+1}^{r,1}) \\ J(\theta_{i+1}^{r,2},\phi_{i+1}^{r,2}) \\ \dots \\ J(\theta_{i+1}^{r,N},\phi_{i+1}^{r,N}) \end{Bmatrix}. \tag{2.20}$$

We define a matrix of front biconical reflectance distribution functions for layer i,

$$\boldsymbol{\rho}_i^f = \begin{Bmatrix} \rho_i^f(\theta_i^{r,1},\phi_i^{r,1};\theta_{i-1}^1,\phi_{i-1}^1) & \rho_i^f(\theta_i^{r,1},\phi_i^{r,1};\theta_{i-1}^2,\phi_{i-1}^2) & \cdots & \rho_i^f(\theta_i^{r,1},\phi_i^{r,1};\theta_{i-1}^N,\phi_{i-1}^N) \\ \rho_i^f(\theta_i^{r,2},\phi_i^{r,2};\theta_{i-1}^1,\phi_{i-1}^1) & \rho_i^f(\theta_i^{r,2},\phi_i^{r,2};\theta_{i-1}^2,\phi_{i-1}^2) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \rho_i^f(\theta_i^{r,N},\phi_i^{r,N};\theta_{i-1}^1,\phi_{i-1}^1) & \cdots & \cdots & \rho_i^f(\theta_i^{r,N},\phi_i^{r,N};\theta_{i-1}^N,\phi_{i-1}^N) \end{Bmatrix} \quad (2.21)$$

so that equation 1.2b becomes

$$\mathbf{J}_i = \boldsymbol{\rho}_i^f \cdot \mathbf{E}_{i-1} \quad (2.22)$$

while for backward-hemisphere incident radiation, equations 1.2c and 1.2d become

$$\mathbf{J}_i = \boldsymbol{\tau}_i^b \cdot \mathbf{E}_{i+1}^r \quad (2.23a)$$

$$\mathbf{I}_i = \boldsymbol{\rho}_i^b \cdot \mathbf{E}_{i+1}. \quad (2.23b)$$

We note that the matrix notation contains the directional information in the position of an element in a vector or matrix. The labeling convention for the reflected (backward-going) radiation has been chosen so that a specularly reflected ray would appear in its reflected vector in the same position as the incident ray; similarly, a specularly transmitted ray would also appear in the same position as its incident ray. With this convention, the specular analogs to the biconical transmittance and reflectance matrices (which will be defined below) will have non-zero elements only on their diagonals. This convention is not necessary, but will prove convenient.

We could next proceed to define $\boldsymbol{\Lambda}$ matrices for the reflected rays; however, with the choice of labeling the definitions would coincide with equation 2.8. In fact, the $\boldsymbol{\Lambda}$ matrices are simply geometrical weighting factors; they depend on the partitioning of the solid angle and on the choice of the underlying basis of directions, but not on the properties of layers. Since our choice of basis is the same for all layers and for both left-moving and right-moving radiation (due to the different coordinate systems to which these are referred), the $\boldsymbol{\Lambda}$ matrices are independent of layer and hemisphere. Thus, the layer notation that we have hitherto added to the $\boldsymbol{\Lambda}$ matrices will be dropped, and no superscript referring to hemisphere will be introduced. The defining equations relating irradiance to radiance then become

$$\mathbf{E}_{i-1} = \boldsymbol{\Lambda} \cdot \mathbf{I}_{i-1} \quad (2.24a)$$

$$\mathbf{E}_{i+1}^r = \boldsymbol{\Lambda} \cdot \mathbf{J}_{i+1} \quad (2.24b)$$

With these definitions equation 2.16 is approximated by the matrix equation

$$\mathbf{M}^{(1)} = \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}_{\mathbf{i-1}}^b \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}_{\mathbf{i}}^f, \quad (2.25a)$$

while 2.17 becomes

$$\mathbf{M}^{(n+1)} = \mathbf{M}^{(1)} \cdot \mathbf{M}^{(n)} \tag{2.25b}$$

and expression 2.18 becomes

$$\left(\mathbf{1} + \sum_{n=1}^{\infty} \mathbf{M}^{(n)}\right) \cdot \mathbf{\Lambda} \cdot \boldsymbol{\tau}_{i-1}^{f}. \tag{2.26}$$

It is shown in Appendix 1 that the quantity in parentheses is simply $\left(\mathbf{1} - \mathbf{M}^{(1)}\right)^{-1}$. Thus, the correct extension of equation 2.10 for the front transmission of a pair of layers, including the interreflections between them is

$$\mathbf{T}_{2\{i-1,i\}}^{f} = \tau_i^f \cdot \left(\mathbf{1} - \mathbf{\Lambda} \cdot \boldsymbol{\rho}_{i-1}^{b} \cdot \mathbf{\Lambda} \cdot \boldsymbol{\rho}_i^f\right)^{-1} \cdot \mathbf{\Lambda} \cdot \tau_{i-1}^{f}. \tag{2.27}$$

Optical Property Matrices for Multilayer Systems

If we consider the general situation of optical propagation through a pair of adjacent layers i-1 and i within an optical system, it is clear that we must consider radiation incident from both forward-going and backward-going hemispheres, as indicated in Figure 6(A), since if layer i is not the last layer in the system, even for a situation in which radiation is incident only on the front side (forward-going), transmitted radiation may be reflected back from other layers downstream. We can write the outgoing radiances indicated in the figure as

$$\mathbf{I}_i = \mathbf{T}_{2\{i-1,i\}}^{f} \cdot \mathbf{E}_{i-2} + \mathbf{R}_{2\{i-1,i\}}^{b} \cdot \mathbf{E}_{i+1}^{r} \tag{3.1a}$$

and

$$\mathbf{J}_i = \mathbf{R}_{2\{i-1,i\}}^{f} \cdot \mathbf{E}_{i-2} + \mathbf{T}_{2\{i-1,i\}}^{b} \cdot \mathbf{E}_{i+1}^{r} \tag{3.1b}$$

in terms of the front and back two-layer (sub)system transmittance and reflectance matrices, as indicated in Figure 6(B) and (C). We have already derived the equation for the 2-layer front transmittance in Section 2,

$$\mathbf{T}_{2\{i-1,i\}}^{f} = \tau_i^f \cdot \left(\mathbf{1} - \mathbf{\Lambda} \cdot \boldsymbol{\rho}_{i-1}^{b} \cdot \mathbf{\Lambda} \cdot \boldsymbol{\rho}_i^f\right)^{-1} \cdot \mathbf{\Lambda} \cdot \tau_{i-1}^{f} \tag{3.2a}$$

and examination of the diagrams in Figure 6(B) and (C) with reference to the scheme for applying matrices indicated in Figures 3 and 5 allows us to write down immediately the expressions for the other subsystem matrices:

$$\mathbf{R}_{2\{i-1,i\}}^{f} = \boldsymbol{\rho}_{i-1}^{f} + \tau_{i-1}^{b} \cdot \left(\mathbf{1} - \mathbf{\Lambda} \cdot \boldsymbol{\rho}_i^f \cdot \mathbf{\Lambda} \cdot \boldsymbol{\rho}_{i-1}^{b}\right)^{-1} \cdot \mathbf{\Lambda} \cdot \boldsymbol{\rho}_i^f \cdot \mathbf{\Lambda} \cdot \tau_{i-1}^{f} \tag{3.2b}$$

$$\mathbf{T}_{2\{i-1,i\}}^{b} = \tau_{i-1}^{b} \cdot \left(\mathbf{1} - \mathbf{\Lambda} \cdot \boldsymbol{\rho}_i^f \cdot \mathbf{\Lambda} \cdot \boldsymbol{\rho}_{i-1}^{b}\right)^{-1} \cdot \mathbf{\Lambda} \cdot \tau_i^b \tag{3.2c}$$

$$\mathbf{R}^{b}_{2,\{i-1,i\}} = \boldsymbol{\rho}^{b}_{i} + \tau^{f}_{i} \cdot \left(\mathbf{1} - \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{b}_{i-1} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{f}_{i}\right)^{-1} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{b}_{i-1} \cdot \boldsymbol{\Lambda} \cdot \tau^{b}_{i} \tag{3.2d}$$

These relations can immediately be applied to an arbitrary system by considering layer i to be the n+1st layer and layer i-1 to be a subsystem composed of the preceding n layers:

$$\mathbf{T}^{f}_{n+1\{1,n+1\}} = \tau^{f}_{n+1} \cdot \left(\mathbf{1} - \boldsymbol{\Lambda} \cdot \mathbf{R}^{b}_{n\{1,n\}} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{f}_{n+1}\right)^{-1} \cdot \boldsymbol{\Lambda} \cdot \mathbf{T}^{f}_{n\{1,n\}} \tag{3.3a}$$

$$\mathbf{R}^{f}_{n+1,\{1,n+1\}} = \mathbf{R}^{f}_{n\{1,n\}} + \mathbf{T}^{b}_{n\{1,n\}} \cdot \left(\mathbf{1} - \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{f}_{n+1} \cdot \boldsymbol{\Lambda} \cdot \mathbf{R}^{b}_{n\{1,n\}}\right)^{-1} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{f}_{n+1} \cdot \boldsymbol{\Lambda} \cdot \mathbf{T}^{f}_{n\{1,n\}} \tag{3.3b}$$

$$\mathbf{T}^{b}_{n+1,\{1,n+1\}} = \mathbf{T}^{b}_{n\{1,n\}} \cdot \left(\mathbf{1} - \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{f}_{n+1} \cdot \boldsymbol{\Lambda} \cdot \mathbf{R}^{b}_{n\{1,n\}}\right)^{-1} \cdot \boldsymbol{\Lambda} \cdot \tau^{b}_{n+1} \tag{3.3c}$$

$$\mathbf{R}^{b}_{n+1,\{1,n+1\}} = \boldsymbol{\rho}^{b}_{n+1} + \tau^{f}_{n+1} \cdot \left(\mathbf{1} - \boldsymbol{\Lambda} \cdot \mathbf{R}^{b}_{n\{1,n\}} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\rho}^{f}_{n+1}\right)^{-1} \cdot \boldsymbol{\Lambda} \cdot \mathbf{R}^{b}_{n\{1,n\}} \cdot \boldsymbol{\Lambda} \cdot \tau^{b}_{n+1} \tag{3.3d}$$

By using the first two layers in an n-layer system to form 2-layer subsystem property matrices using equations 3.2 and then repetitively applying equations 3.3 to add the next adjacent layer, the system property matrices for the complete system may be derived.

The directional-hemispherical transmittance or reflectance of a given layer is computed by summing the particular column of the layer property matrix over the outgoing solid angle, with a $\cos(\theta)$ weighting to account for the projection of a given element of surface area in each outgoing direction. If this process is repeated for each column of the matrix the result is a row vector of directional-hemispherical layer transmittances for each of the basis directions. This process can be included in the matrix formalism by defining auxiliary column and row vectors $\mathbf{u}$ and $\mathbf{u}^{\mathrm{T}}$ as follows:

$$\mathbf{u} = \begin{Bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{Bmatrix} \tag{3.4a}$$

$$\mathbf{u}^{\mathrm{T}} = \{1 \quad 1 \quad \ldots \quad 1\} \tag{3.4b}$$

and defining the directional-hemispherical row vectors for layer i by

$$\boldsymbol{\tau}^{f,DH}_{i} = \mathbf{u}^{\mathrm{T}} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\tau}^{f}_{i}, \tag{3.5}$$

with similar definitions for back transmittance and front and back reflectance, i.e., the addition of the superscript DH to the property matrix symbol indicates that it is a row vector of directional-hemispherical transmittances formed by left-multiplication of the property matrix by $\mathbf{u}^{\mathrm{T}} \cdot \boldsymbol{\Lambda}$. One can form an analogous column vector of hemispherical-directional properties by right-multiplication of the matrix by $\mathbf{\Lambda} \cdot \mathbf{u}$, as indicated in Appendix 1, but we will not have occasion to use these.

We then define layer front and back absorption row vectors using Kirchhoff's Law:

$$\boldsymbol{\alpha}_i^f = \mathbf{1} - \boldsymbol{\tau}_i^{f,DH} - \boldsymbol{\rho}_i^{f,DH} \tag{3.6a}$$

and

$$\boldsymbol{\alpha}_i^b = \mathbf{1} - \boldsymbol{\tau}_i^{b,DH} - \boldsymbol{\rho}_i^{b,DH} \tag{3.6b}$$

These two vectors correspond to the absorption of layer i taken in isolation (e.g., as measured in the scanning radiometer). However, for the same layer as the ith layer of an M layer system, the situation is complicated by multiple reflections between the ith layer and the upstream and downstream layers, as indicated in Figure 7(A). In addition to an incident ray that is transmitted through the first i-1 layers and absorbed in layer i, one must also consider rays which (1) are reflected from layer i, rereflected from an upstream layer, and absorbed in i; (2) are transmitted through i, reflected from a downstream layer, incident on the back side of i and absorbed; (3) transmitted through i, reflected from a downstream layer, transmitted again through i, rereflected by an upstream layer, and absorbed in i; together with all possible higher-order combinations of these processes. In general, then, it can be seen that the front and back layer/system absorptances, $\mathbf{A}_{i;M}^f$ and $\mathbf{A}_{i;M}^b$, for layer i as the ith layer of an M-layer system each depend on both the isolated-layer absorption vectors $\boldsymbol{\alpha}_i^f$ and $\boldsymbol{\alpha}_i^b$.

We can compute these layer/system absorptances by using the two alternative decompositions of the M-layer system into a pair of subsystems shown in Figure 7(B) and (C). For radiation incident on the front of layer i we decompose the system as shown in (B), while for radiation incident on the back of layer i we decompose it as shown in (C). The resulting expressions for the system/layer absorption vectors are

$$\begin{aligned}\mathbf{A}_{i;M}^f = {} & \boldsymbol{\alpha}_i^f \cdot (\mathbf{1} - \mathbf{\Lambda} \cdot \mathbf{R}_{i-1,\{1,i-1\}}^b \cdot \mathbf{\Lambda} \cdot \mathbf{R}_{(M-i+1),\{i,M\}}^f)^{-1} \cdot \mathbf{\Lambda} \cdot \mathbf{T}_{i-1,\{1,i-1\}}^f \\ & + \boldsymbol{\alpha}_i^b \cdot (\mathbf{1} - \mathbf{\Lambda} \cdot \mathbf{R}_{(M-i),\{i+1,M\}}^f \cdot \mathbf{\Lambda} \cdot \mathbf{R}_{i,\{1,i\}}^b)^{-1} \cdot \mathbf{\Lambda} \cdot \mathbf{R}_{(M-i),\{i+1,M\}}^f \cdot \mathbf{\Lambda} \cdot \mathbf{T}_{i,\{1,i\}}^f\end{aligned} \tag{3.7a}$$

$$\begin{aligned}\mathbf{A}_{i;M}^b = {} & \boldsymbol{\alpha}_i^b \cdot (\mathbf{1} - \mathbf{\Lambda} \cdot \mathbf{R}_{(M-i),\{i+1,M\}}^f \cdot \mathbf{\Lambda} \cdot \mathbf{R}_{i,\{1,i\}}^b)^{-1} \cdot \mathbf{\Lambda} \cdot \mathbf{T}_{M-i,\{i+1,M\}}^b \\ & + \boldsymbol{\alpha}_i^f \cdot (\mathbf{1} - \mathbf{\Lambda} \cdot \mathbf{R}_{i-1,\{1,i-1\}}^b \cdot \mathbf{\Lambda} \cdot \mathbf{R}_{(M-i+1),\{i,M\}}^f)^{-1} \cdot \mathbf{\Lambda} \cdot \mathbf{R}_{i-1,\{1,i-1\}}^b \cdot \mathbf{\Lambda} \cdot \mathbf{T}_{M-i+1,\{i,M\}}^b\end{aligned} \tag{3.7b}$$

The elements of the row vector $\mathbf{A}_{i;M}^f$ are the layer front absorptions of eqn. 1.1 evaluated at the incident angles corresponding to points on the coordinate net in Figure 2:

$$\mathbf{A}_{i;M}^f = \left\{ A_{fi}(\theta_0^1, \phi_0^1) \quad A_{fi}(\theta_0^2, \phi_0^2) \quad \ldots \quad A_{fi}(\theta_0^N, \phi_0^N) \right\}. \tag{3.8}$$

The system directional-hemispherical front transmittance row vector,

$$\mathbf{T}_M^{f,DH} = \mathbf{u}^{\mathrm{T}} \cdot \mathbf{\Lambda} \cdot \mathbf{T}_{M,\{1,M\}}^f, \tag{3.9}$$

is similarly an array of the system front hemispherical transmittance function values on the coordinate net:

$$\mathbf{T}_M^{f,DH} = \left\{ T_{fH}(\theta_0^1,\phi_0^1) \quad T_{fH}(\theta_0^2,\phi_0^2) \quad \dots \quad T_{fH}(\theta_0^N,\phi_0^N) \right\}. \tag{3.10}$$

We thus see that both of the solar-optical quantities necessary to the computation of the solar heat gain coefficient may be calculated from the layer properties for systems of any degree of complexity.

Specular Layers

Specular layers represent a special case for which the generality of the above computations, with their implicit integrations over solid angle, is unnecessary. For the integral language of Paper I specular layers always have property matrices containing the mathematically well-known delta distribution, which plays the role of an identity operator in integral transformations:

$$\int \delta(x',x) f(x') dx' = f(x) \tag{4.1}$$

for any function f and any region of integration including x (otherwise the integral is zero).

Physically, this means that a ray incident on a given layer i with a given direction must result in a ray incident on the next (or reflected back to the previous) layer with the same (or the specularly reflected) direction, with the intensity multiplied by the specular transmittance (reflectance) corresponding to that direction. In our matrix language that means

$$\mathbf{E}_i = \mathbf{\Lambda} \cdot \boldsymbol{\tau}_i \cdot \mathbf{E}_{i-1} = \begin{pmatrix} \tau_{(S)i}^{(1)} & 0 & \dots & 0 \\ 0 & \tau_{(S)i}^{(2)} & 0 & \dots \\ \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \tau_{(S)i}^{(N)} \end{pmatrix} \cdot \mathbf{E}_{i-1}, \tag{4.2a}$$

where the quantities on the diagonal of the matrix are the specular transmittances of the specular layer i for each incident (and outgoing) direction, or, in terms of the individual matrix elements,

$$(E_i)_k = \sum_{l,m} (\Lambda)_m \cdot (\tau_i)_{ml} \cdot (E_{i-1})_l = \left(\tau_{(S)i}\right)_k \cdot (E_{i-1})_k. \tag{4.2b}$$

This means that the transmittance matrix for the specular layer must be given by

$$\boldsymbol{\tau}_i = \mathbf{\Lambda}^{-1} \cdot \begin{pmatrix} \tau_{(S)i}^{(1)} & 0 & \dots & 0 \\ 0 & \tau_{(S)i}^{(2)} & 0 & \dots \\ \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \tau_{(S)i}^{(N)} \end{pmatrix} \tag{4.3a}$$

or

$$\boldsymbol{\tau}_i = \begin{pmatrix} \frac{\tau_{(S)i}^{(1)}}{\Lambda^{(1)}} & 0 & \dots & 0 \\ 0 & \frac{\tau_{(S)i}^{(2)}}{\Lambda^{(2)}} & 0 & \dots \\ \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\tau_{(S)i}^{(N)}}{\Lambda^{(N)}} \end{pmatrix}. \tag{4.3b}$$

With this definition for front and back transmittance matrices, and the analogous ones for front and back reflectance, utilizing the appropriate directional specular properties for the computation of the diagonal matrix elements, specular layers may be included in the calculation scheme on an equal footing with non-specular layers.

An Example Calculation

As an example of the calculation method, we consider a simplified example of a perfectly-diffusing interior shade with a clear double-glazed window. We shall suppose that the shade is a Lambertian diffuser in both transmittance and reflectance, with a hemispherical transmittance of 0.21 independent of incident angle, and an angle-independent hemispherical reflectance of 0.62. This is a strong (and possibly unrealistic) assumption, since even for a featureless shade one can assume at best reflection symmetry through the plane of incidence for incident angles other than normal; it is made to simplify the calculation. The calculation can be visualized by referring to Figure 7 for the case i=2, M=3.

With this assumption the calculation is independent of the angle ϕ in both the incident and outgoing directions. As a result, each ray of the angular networks shown in Figure 2 will have identical values. We can thus suppress the azimuthal dependence and deal with 7X7 dimensional matrices in the variable θ, instead of the 145X145 dimensional matrices that a more complicated case would require. We take as our angular basis the points

$$\theta_l = \{0^\circ \quad 15^\circ \quad 30^\circ \quad 45^\circ \quad 60^\circ \quad 75^\circ \quad 86.25^\circ\}, \tag{5.1}$$

with the corresponding propagation matrix

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0.054 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.407 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.704 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.813 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.704 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.407 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.054 \end{pmatrix}. \tag{5.2}$$

In defining solid angle segments, boundaries between angular regions were taken to be the midpoint between the values in equation 5.1 except for the final bin, which extends from 82.5° to 90°. The actual calculation was carried out using more significant figures than will generally be presented here, so that in reproducing the calculation the reader must allow for round-off error.

Under the assumptions, the bidirectional transmittance of the shade will be simply τ^{DH}/π and the bidirectional reflectance ρ^{DH}/π, independent of angle. Hence the layer property matrices for this layer (number 3 in the system) will be

$$\boldsymbol{\tau}_3^f = \boldsymbol{\tau}_3^b = \begin{pmatrix} 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \\ 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 & 0.067 \end{pmatrix} \tag{5.3a}$$

and

$$\boldsymbol{\rho}_3^f = \boldsymbol{\rho}_3^b = \begin{pmatrix} 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \\ 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 & 0.198 \end{pmatrix}. \tag{5.3b}$$

For the glass layers we use published (Rubin 1985) transmittance and reflectance data for 2.5mm thick clear glass,

$$\tau_S(\theta_l) = \{0.856 \quad 0.855 \quad 0.850 \quad 0.832 \quad 0.773 \quad 0.532 \quad 0.152\} \tag{5.4a}$$

and

$$\rho_S(\theta_l) = \{0.077 \quad 0.077 \quad 0.079 \quad 0.093 \quad 0.146 \quad 0.387 \quad 0.818\}, \tag{5.4b}$$

from which we can construct the specular layer matrices:

$$\boldsymbol{\tau}_1^f = \boldsymbol{\tau}_1^b = \boldsymbol{\tau}_2^f = \boldsymbol{\tau}_2^b = \begin{pmatrix} 15.99 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.21 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.02 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.31 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.84 \end{pmatrix} \tag{5.5a}$$

and

$$\boldsymbol{\rho}_1^f = \boldsymbol{\rho}_1^b = \boldsymbol{\rho}_2^f = \boldsymbol{\rho}_2^b = \begin{pmatrix} 1.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.189 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.112 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.114 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.207 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.952 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15.28 \end{pmatrix}. \qquad (5.5b)$$

From these we can now carry out the calculation. We first calculate the system properties of the double glazing sub-system (which of course is front-back symmetric):

$$\mathbf{T}_{2\{1,2\}}^f = \mathbf{T}_{2\{1,2\}}^b = \begin{pmatrix} 13.77 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.81 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.03 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.859 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.867 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.819 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.30 \end{pmatrix} \qquad (5.6a)$$

$$\mathbf{R}_{2\{1,2\}}^f = \mathbf{R}_{2\{1,2\}}^b = \begin{pmatrix} 2.50 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.329 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.194 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.194 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.334 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16.35 \end{pmatrix}, \qquad (5.6b)$$

using equations 3.2. In this we have used the subsystem interreflectance matrix,

$$\left(\mathbf{1} - \boldsymbol{\Lambda}\cdot\boldsymbol{\rho}_1^b\cdot\boldsymbol{\Lambda}\cdot\boldsymbol{\rho}_2^f\right)^{-1} = \begin{pmatrix} 1.01 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.02 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.18 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.02 \end{pmatrix}, \qquad (5.7)$$

from which it is apparent that multiple reflectances for this subsystem are important only at large angles. From equations 3.3 we can then obtain the system transmittance matrix,

$$\mathbf{T}_3^f = \begin{pmatrix} 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \\ 0.058 & 0.058 & 0.057 & 0.055 & 0.048 & 0.026 & 0.005 \end{pmatrix}, \tag{5.8}$$

from which the system directional-hemispherical front transmittance is computed using equation 3.9:

$$\mathbf{T}_3^{f,DH} = \begin{pmatrix} 0.182 & 0.181 & 0.179 & 0.172 & 0.150 & 0.082 & 0.017 \end{pmatrix}. \tag{5.9a}$$

Similarly, the system directional-hemispherical front reflectance may also be computed, although it is not required for the solar heat gain coefficient calculation:

$$\mathbf{R}_3^{f,DH} = \begin{pmatrix} 0.470 & 0.470 & 0.468 & 0.477 & 0.514 & 0.668 & 0.907 \end{pmatrix}. \tag{5.9b}$$

For calculation of the layer directional absorptances one also needs the subsystem reflectance matrix for the pairing of layers 2 and 3:

$$\mathbf{R}_{2\{2,3\}}^f = \begin{pmatrix} 1.60 & 0.160 & 0.159 & 0.155 & 0.144 & 0.10 & 0.03 \\ 0.160 & 0.349 & 0.159 & 0.155 & 0.144 & 0.10 & 0.03 \\ 0.159 & 0.159 & 0.270 & 0.154 & 0.143 & 0.10 & 0.03 \\ 0.155 & 0.155 & 0.154 & 0.265 & 0.140 & 0.10 & 0.03 \\ 0.144 & 0.155 & 0.143 & 0.140 & 0.338 & 0.09 & 0.03 \\ 0.10 & 0..10 & 0.10 & 0.10 & 0.09 & 1.014 & 0.02 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.02 & 15.29 \end{pmatrix}. \tag{5.10}$$

The system/layer front directional absorptances for the three layers can then be calculated, first using equations 3.6 to calculate the directional absorptions for the individual (isolated) layers from equations 5.3 and 5.4:

$$\boldsymbol{\alpha}_1^f = \boldsymbol{\alpha}_1^b = \boldsymbol{\alpha}_2^f = \boldsymbol{\alpha}_2^b = \begin{pmatrix} 0.067 & 0.068 & 0.071 & 0.075 & 0.081 & 0.081 & 0.03 \end{pmatrix} \tag{5.11a}$$

$$\boldsymbol{\alpha}_3^f = \boldsymbol{\alpha}_3^b = \begin{pmatrix} 0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 \end{pmatrix} \tag{5.11b}$$

then using equations 3.7 to get the system-dependent layer absorptances:

$$\mathbf{A}^{f}_{1;3} = \begin{pmatrix} 0.103 & 0.104 & 0.107 & 0.111 & 0.117 & 0.115 & 0.044 \end{pmatrix} \tag{5.12a}$$

$$\mathbf{A}^{f}_{2;3} = \begin{pmatrix} 0.102 & 0.102 & 0.104 & 0.105 & 0.100 & 0.070 & 0.018 \end{pmatrix} \tag{5.12b}$$

$$\mathbf{A}^{f}_{3;3} = \begin{pmatrix} 0.143 & 0.143 & 0.141 & 0.136 & 0.119 & 0.065 & 0.014 \end{pmatrix}. \tag{5.12c}$$

These are system-dependent quantities because of the multiple reflections. The calculated system properties are shown in Figure 8.

Conclusion

Beginning with the equations derived in Paper I, we have seen how the overall system solar-optical properties of a multilayer fenestration with one or more non-specular layers can be calculated from the individual layer bi-directional properties. It has been shown that multiple reflections between layers can be represented with an inverse matrix, in complete analogy to the formulas used for specular multilayer systems. New features appearing in the calculation were diagonal propagation matrices, which are layer-independent geometrical quantities. Scalar quantities in the specular treatment become matrices, and of course the ordering of terms in products becomes significant; a notation was chosen in which propagation from front to back through a system corresponds to moving from right to left in the corresponding matrix expression.

We have seen that a pair of adjacent layers can be combined into a subsystem, and by repeatedly considering pairings between a subsystem and an adjacent layer, system properties may be built up recursively in a manner that automatically includes all interreflections between layers. We have seen how to calculate the system directional-hemispherical transmittance from the system bi-directional transmittance, and how to use the same procedures of combination into subsystems to determine the layer absorptances. In a simplified example calculation, we have seen how the calculation scheme is utilized to determine all of the solar-optical system properties.

In the example, extreme symmetry assumptions were made to reduce the calculation to a tractable one. For more realistic cases where less symmetry can be assumed a priori the calculation uses very large matrices for which computer calculation is the only feasible method. The equations have been incorporated into a set of computer programs (called TRA) that was used in the to carry out the system property determinations for the project.

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References

Klems, J. H. (1993A). "A New Method for Predicting the Solar Heat Gain of Complex Fenestration Systems: I. Overview and Derivation of the Matrix Layer Calculation." ASHRAE, to be published. :

Nicodemus, F. E. (1965). "Directional Reflectance and Emissivity of an Opaque Surface." Appl. Opt. **4**(7): 767-773.

Papamichael, K. and F. Winkelmann (1986). Solar-Optical Properties of Multilayer Fenestration Systems. International Daylighting Conference, Long Beach, CA,

Rubin, M. (1985). "Optical Properties of Soda Lime Silica Glasses." Solar Energy Matls. **12**: 275-288.

Appendix 1. Convergence of the Multiple Reflectance Series

Here we present a mathematical proof that the multiple reflectance series of reflectance matrices converges, and hence that the use of an inverse matrix operator in Section 2 is justified. While convergence of the reflectance series is necessary on physical grounds--otherwise energy would not be conserved--it is still desirable to show that for the finite matrix representation chosen this necessary condition is preserved. The proof will rest on the fact that as defined, each column of the matrix representing the product $\mathbf{\Lambda} \cdot \boldsymbol{\rho}$ sums to produce a directional-hemispherical reflectance for the layer, while summing a row produces a hemispherical-directional reflectance:

$$\sum_l \left(\Lambda \rho_i^f\right)_{lm} = r_{i,DH,m}^f \tag{A1.1a}$$

$$\sum_l \left(\Lambda \rho_{i-1}^b\right)_{kl} = r_{i-1,HD,k}^b = r_{i-1,DH,k}^b, \tag{A1.1b}$$

where the second equality in A1.1b follows from time-reversal invariance, i.e., the reflectance property is unchanged if the directions of all incoming and outgoing rays are simultaneously reversed. For each layer surface, we define the quantity r (with appropriate subscripts and superscripts) to be the maximum value of the directional-hemispherical reflectances for that surface. Hence,

$$\sum_l \left(\Lambda \rho_i^f\right)_{lm} \le r_i^f < 1 \tag{A1.2a}$$

$$\sum_l \left(\Lambda \rho_{i-1}^b\right)_{kl} \le r_{i-1}^b < 1, \tag{A1.2b}$$

where the strict inequality on the right follows from the fact that for any non-perfect reflector the directional-hemispherical reflectance is less than one.

The multiple reflection series between layers i-1 and i, indicated by the diagram in Figure 5, is given by

$$1 + \sum_{n=1}^{\infty} \mathbf{M}^{(n)}, \tag{A1.3}$$

where

$$\mathbf{M}^n = \prod^n \left(\mathbf{\Lambda}_{i-1} \boldsymbol{\rho}_{i-1}^b \mathbf{\Lambda}_i \boldsymbol{\rho}_i^f\right), \tag{A1.4}$$

and it is understood that successive factors in the product are formed from right to left. If we consider an arbitrary element M_{km}^n of the matrix $\mathbf{M}^n$ we can see that it has the following form:

$$M_{km}^{n}=\sum_{l=1}^{N}\sum_{p=1}^{N}\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\left(\Lambda_{i}\rho_{i}^{f}\right)_{lp}M_{pm}^{n-1} \tag{A1.5a}$$

for $n>1$, while

$$M_{km}^{1}=\sum_{l=1}^{N}\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\left(\Lambda_{i}\rho_{i}^{f}\right)_{lm}. \tag{A1.5b}$$

By the Schwartz inequality,

$$\sum_{l=1}^{N}\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\left(\Lambda_{i}\rho_{i}^{f}\right)_{lm}\leq\left\{\left(\sum_{l=1}^{N}\left[\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\right]^{2}\right)\left(\sum_{l=1}^{N}\left[\left(\Lambda_{i}\rho_{i}^{f}\right)_{lm}\right]^{2}\right)\right\}^{\frac{1}{2}}, \tag{A1.6}$$

while A1.2 combined with the fact that all of the terms in each series are positive implies that

$$\sum_{l=1}^{N}\left[\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\right]^{2}<\sum_{l=1}^{N}\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\leq r_{i-1}^{b} \tag{A1.7a}$$

and

$$\sum_{l=1}^{N}\left[\left(\Lambda_{i}\rho_{i}^{f}\right)_{lm}\right]^{2}<\sum_{l=1}^{N}\left(\Lambda_{i}\rho_{i}^{f}\right)_{lm}\leq r_{i}^{f}. \tag{A1.7b}$$

Thus

$$M_{lm}^{1}<\left(r_{i-1}^{b}r_{i}^{f}\right)^{\frac{1}{2}} \tag{A1.8}$$

Substituting this relation into A1.5a for $n=2$, we see that

$$\begin{aligned}M_{km}^{2}&<\left(r_{i-1}^{b}r_{i}^{f}\right)^{\frac{1}{2}}\sum_{l=1}^{N}\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\sum_{p=1}^{N}\left(\Lambda_{i}\rho_{i}^{f}\right)_{lp}\\&\leq\left(r_{i-1}^{b}r_{i}^{f}\right)^{\frac{1}{2}}r_{i}^{f}\sum_{l=1}^{N}\left(\Lambda_{i-1}\rho_{i-1}^{b}\right)_{kl}\leq\left(r_{i-1}^{b}r_{i}^{f}\right)^{1+\frac{1}{2}}\end{aligned} \tag{A1.9}$$

Repeating this argument, one can readily prove by mathematical induction that

$$M_{lm}^{n}<\left(r_{i-1}^{b}r_{i}^{f}\right)^{n-\frac{1}{2}}. \tag{A1.10}$$

This means that the series for an arbitrary matrix element in A1.3 is dominated term-by term by a geometric series:

$$1+\sum_{n=1}^{\infty}M_{lm}^{n}<1+\left(r_{i-1}^{b}r_{i}^{f}\right)^{-\frac{1}{2}}\sum_{n=1}^{\infty}\left(r_{i-1}^{b}r_{i}^{f}\right)^{n} \tag{A1.11}$$

and must therefore converge. Since each matrix element of expression A1.3 converges to a finite limit, the matrix expression also converges. We denote the limit of the matrix series by $\mathbf{M}$.

We note that equation A1.5 recast in matrix form means that

$$\mathbf{M}^1 \cdot \mathbf{M}^N = \mathbf{M}^{N+1}. \tag{A1.12}$$

We use this relation to show that the standard argument for summing the geometric series also holds for the interreflectance matrix series

$$\mathbf{1} + \mathbf{M} = \mathbf{1} + \sum_{n=1}^{\infty} \mathbf{M}^{(n)}. \tag{A1.13}$$

If we left-multiply this series by the factor $\mathbf{1} - \mathbf{M}^1$ we obtain

$$\begin{aligned}(\mathbf{1} - \mathbf{M}^1) \cdot \left(\mathbf{1} + \sum_{n=1}^{\infty} \mathbf{M}^{(n)}\right) &= \mathbf{1} - \mathbf{M}^1 + \mathbf{M}^1 - \mathbf{M}^1 \cdot \mathbf{M}^1 + \mathbf{M}^2 \\ &+ \ldots - \mathbf{M}^1 \cdot \mathbf{M}^N + \mathbf{M}^{N+1} + \ldots\end{aligned} \tag{A1.14}$$

It can be seen that because of A1.12, successive pairs of terms in this series cancel, leaving

$$(\mathbf{1} - \mathbf{M}^1) \cdot \left(\mathbf{1} + \sum_{n=1}^{\infty} \mathbf{M}^{(n)}\right) = \mathbf{1} \tag{A1.15}$$

and if we denote the inverse matrix of $\mathbf{1} - \mathbf{M}^1$ by $(\mathbf{1} - \mathbf{M}^1)^{-1}$ and use equation A1.4 we obtain

$$\mathbf{1} + \sum_{n=1}^{\infty} \mathbf{M}^{(n)} = \left(\mathbf{1} - \mathbf{\Lambda}_{i-1} \boldsymbol{\rho}_{i-1}^{b} \mathbf{\Lambda}_i \boldsymbol{\rho}_i^{f}\right)^{-1} \tag{A1.16}$$

This completes the proof.

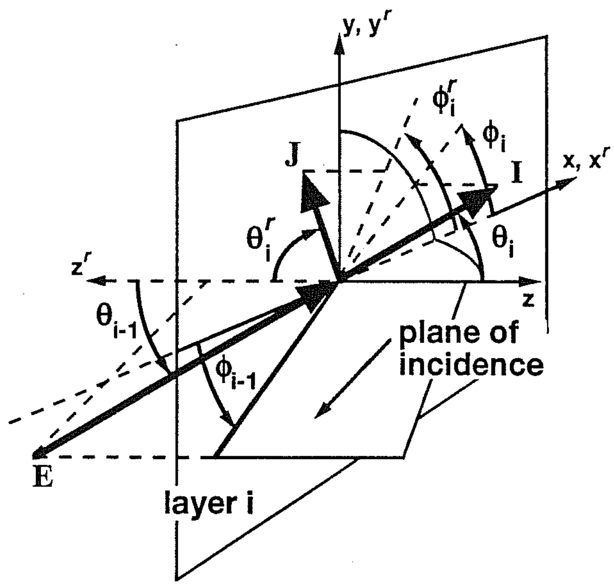

Fig. 1. Definition of the Coordinate Systems for a Layer. The incident irradiance $E(\theta_{i-1}, \phi_{i-1})$ and the forward-going (i.e., transmitted) radiance $I(\theta_i, \phi_i)$ are described in the xyz coordinate system, while the backward-going (i.e., reflected) radiance $J(\theta_i^r, \phi_i^r)$ is described in the reflected coordinate system $x^r y^r z^r$, which is left-handed. All quantities with a superscript r refer to the latter. For the general case I and J may have any direction in their respective hemispheres, as indicated. For specular radiation both I and J would lie in the plane of incidence, with $\theta_i^r = \theta_i$. Note that the forward and backward coordinate systems are related by a reflection through the xy plane, so that in that plane they represent the same two spatial axes viewed from opposite sides.

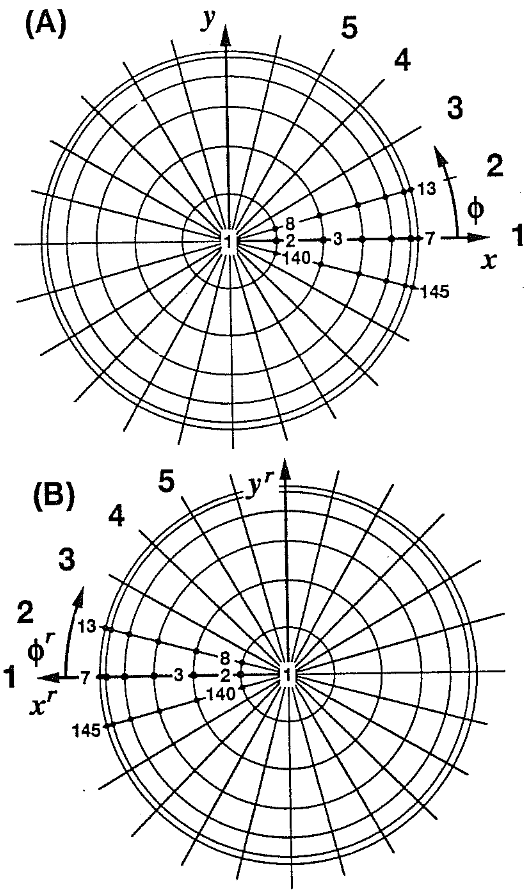

Fig. 2. Angle Coordinates for Incident, Transmitted and Reflected Rays. (A) Coordinates for incident and forward-going radiation. The angles are refered to the xyz coordinate system of Fig. 1.2; in this figure the z axis is perpendicular the plane and points toward the viewer. The numbers indicate the ordering of directions in constructing vectors and matrices. (B) Coordinates for backward-going radiation. The angles are refered to the $x^r y^r z^r$ coordinate system in Fig. 1.2; the z^r axis points out of the plane of the figure.

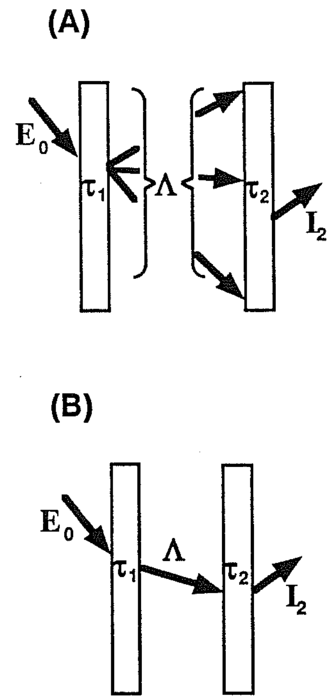

Fig. 3. Schematic Representations of Transmission Through a Pair of Generalized Layers. (A) Incident radiation E_0 in a particular direction is scattered in the first layer, producing a distribution of output rays. The function of the Λ matrix in converting the outgoing radiance for each ray into the incoming irradiance at the next layer is indicated. Of all the outgoing rays from the second layer, the radiance I_2 in a particular direction is specified. (B) The more abbreviated schematic which will be used for the same process, indicating an incident irradiance E_0, a transmission τ_1, a propagation Λ, a transmission τ_2, and an outgoing radiance I_2.

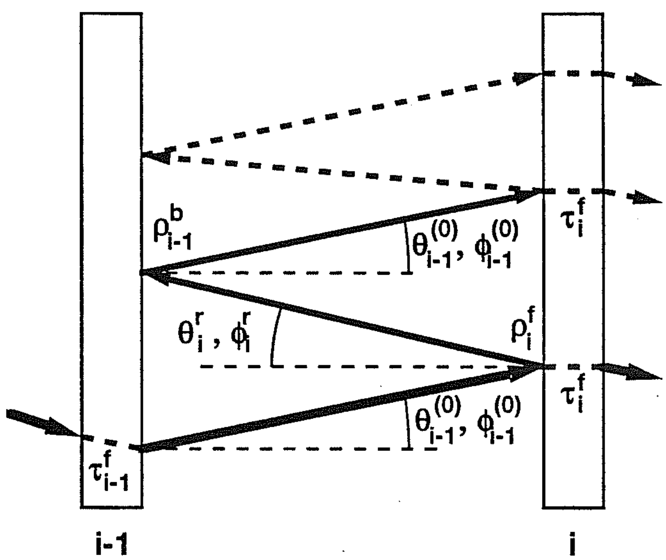

Fig. 4. Multiple Reflection Corrections to the Transmission Through a Pair of Layers. As in Figure 3, each indicated backward-going ray is in reality a set of rays distributed over the appropriate pair of outgoing angles, which are given in the backward-going coordinate system (hence the superscript r). The superscript (0) on the forward-going rays indicates that a particular incident direction for layer i has been selected from among the distribution of rays reflected off the front of layer i-1.

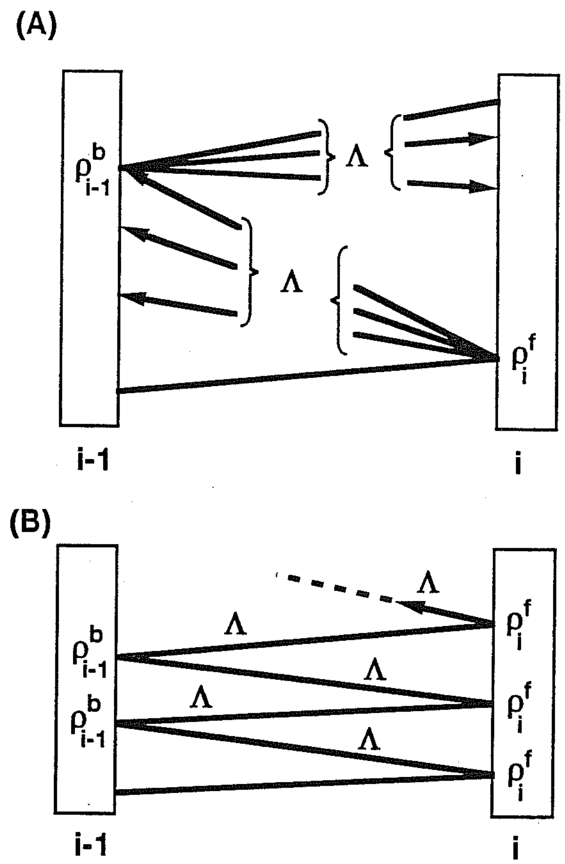

Fig. 5. Schematic Representations of Interreflections Between Adjacent Layers. (A) A representation of the physical process, showing that the first reflection at layer i produces a distribution of reflected rays, and that each of these rays produces another distribution on reflection by layer i-1. The role of the propagation matrix in converting each ray from an outgoing radiance to an incoming irradiance at the next layer is indicated. (B) The more abbreviated notation for the same process. In this notation, single paths between layers symbolize entire distributions of intermediate rays and serve to indicate the sequence of scattering events (and thus the order of matrices in the calculation) rather than the physical path of intermediate rays.

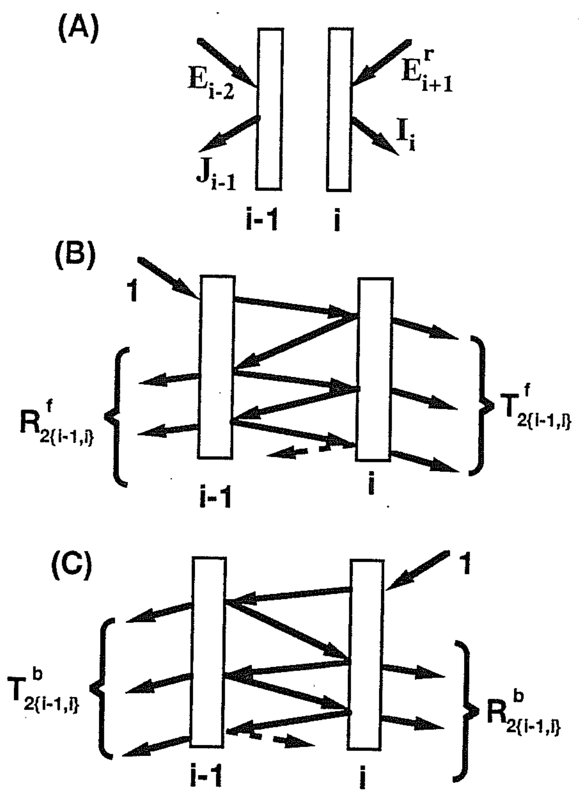

Fig. 6 Diagrammatic treatment of an interior pair of layers, for which radiation may be incident on both front and back (A), showing how the pair of layers may be composed into front (B) and back (C) subsystem transmittances.

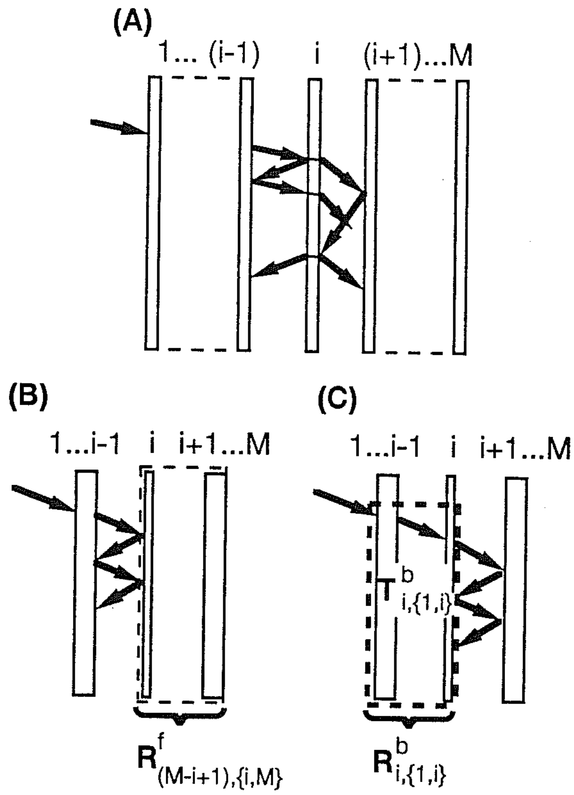

Fig. 7. Diagrammatic representation of the calculation of layer absorptions. As indicated in (A) the situation is complicated by the fact that rays reflecting from upstream or downstream layers may be either reflected or transmitted by the layer of interest, in addition to being absorbed. As shown in (B) and (C), this problem may be handled by decomposing the system into two subsystems in one way for forward-going radiation (B) and in a different way for backward-going radiation (C).

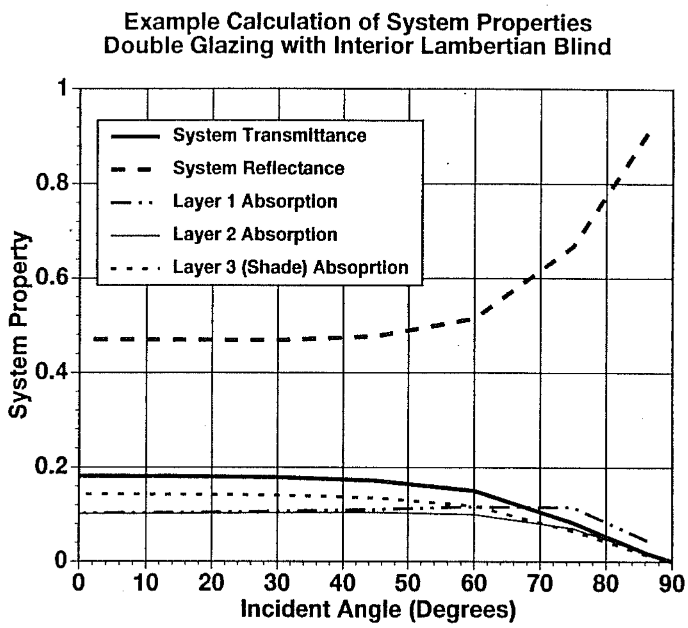

Fig. 8. Calculated system solar-optical properties for the example system in section 5.